method [10]. Among the advantages of the method of estimates, we should make mention of the following: first, the simplicity in finding subsequent expansions in powers of $1/\theta_0$, second, a knowledge of the range over which the exact solution is applicable; and the simplicity of generalization to a heat release function which is different from the Arrhenius function. Preliminary results of this paper as applicable to laminar flames are discussed in [11].

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NONLINEAR ANALYSIS OF THE FLOW INITIATED BY THE SUDDEN MOTION OF A WEDGE

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Certain self-similar problems involving the sudden motion of a wedge which were treated in the linear approximation in [1-3] are studied by the method of matched asymptotic expansions. The nature of the wave boundary of the perturbed region is determined. Second-approximation solutions are constructed which describe flows behind weak shock fronts propagating in a stationary gas and behind fronts of weak discontinuity lines propagating by known uniform flows. A boundary-value problem is formulated whose solution describes, in first approximation, flows in the neighborhoods of points of interaction of the fronts. The existence of similarity rules of flows in these nieghborhoods is estimated. An approximate solution of the problems is given.

§1. Let us consider the flow of a stationary ideal polytropic gas arising from the sudden motion of an infinite wedge with constant velocity W_0 in the negative Ox direction. The parameters of this self-similar problem are the Mach number of the wedge $M_0 = W_0/a_0$, the adiabatic exponent of the gas γ, and the angles α_1, α_2 between the edges of the wedge of the Ox axis. The following cases are examined; a) a wedge with arbitrary vertex angle moving at low velocity $M_0 \ll 1$; b) a thin wedge $\alpha_j = \alpha \ll 1$ moving with subsonic velocity; c) a thin wedge moving with supersonic velocity (Fig. 1a-b, c, respectively). In the latter case the condition $\alpha M_0 \ll 1$ must be satisfied.

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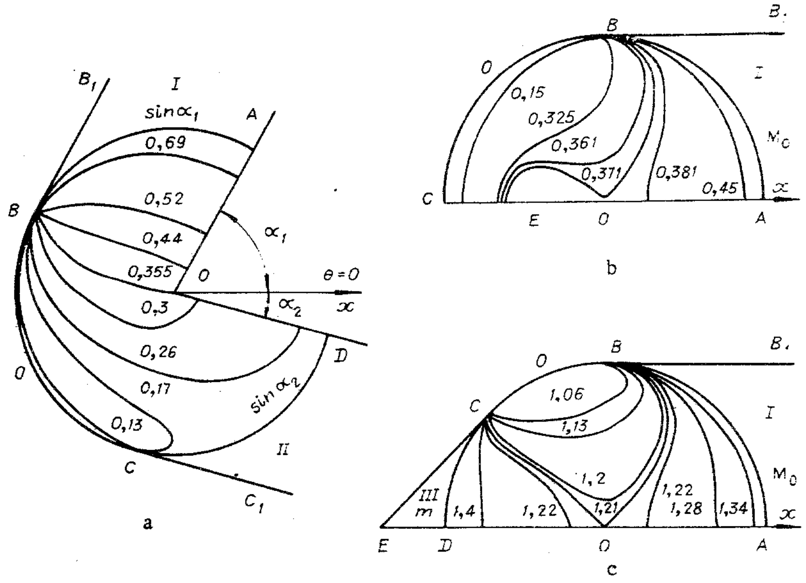

Fig. 1

In regions I-III of Fig. 1 outside the zone OABCDO of the effect of an edge of the wedge (perturbed region) we have uniform flows whose parameters are known [1, 4] (p is the pressure, U is the velocity of a plane shock front, and the subscripts 0 and j correspond to the parameters of the stationary gas and uniform flows):

$$(p_j - p_0)/\gamma p_0 = [2/(\gamma + 1)][(U_j/a_0)^2 - 1], \tag{1.1}$$

$$U_j/a_0 = [(\gamma + 1)/4]M_0\sin\alpha_j + \sqrt{1 + \{[(\gamma + 1)/4]M_0\sin\alpha_j\}^2}.$$

The boundary ABCD and the nonuniform flow parameters in the perturbed region must be determined.

Henceforth, either M_0 or α will be considered as the small parameter of the problem ε. According to (1.1) the strength of plane shock waves is of order ε in the cases considered, and, consequently, the flow in the perturbed region is irrotational to order ε^2 inclusively.

The expressions for the velocity potential f and the Lagrange–Cauchy equation in self-similar variables in polar coordinates r, θ have the form [5-8]

$$\begin{aligned}(1 - r^2) f_{rr} + (1/r) f_r + (1/r^2) f_{\theta\theta} = (\gamma - 1)(f - rf_r + (1/2) f_r^2 + \\ + (1/2r^2) f_\theta^2)(f_{rr} + (1/r) f_r + (1/r^2) f_{\theta\theta}) + (f_r^2 - 2rf_r) f_{rr} + \\ + (2/r^2) f_\theta (f_\theta - rf_{r\theta}) + (1/r^4) f_\theta^2 f_{\theta\theta} + (2/r^2) f_r f_\theta f_{r\theta} - (1/r^3) f_r f_\theta^2;\end{aligned} \tag{1.2}$$

$$a_*^2 = (1 + \gamma P)^{(\gamma-1)/\gamma} = 1 - (\gamma - 1)(f - rf_r + (1/2) f_r^2 + (1/2r^2) f_\theta^2), \tag{1.3}$$

where $f = \Phi / a_0^2 t$; $P = (p - p_0)/\gamma p_0$; $r\cos\theta = x/a_0 t$; $r\sin\theta = y/a_0 t$ [Φ(x, y, t) is the velocity potential in the stationary coordinate system].

The unknown boundary of the perturbed region is the aggregate of weak shock fronts and weak discontinuity lines propagating in the stationary gas or by known uniform flows.

We write the differential equations of the shock fronts $r = k(\theta)$, the weak discontinuity lines $r = r_*(\theta)$ propagating by uniform flows, and the conditions on them in the form [5-8]

$$f_r - f_{jr} = \frac{2}{\gamma + 1}\left[\frac{D_2}{1 + k'^2/k^2} - \frac{a_{j*}^2}{D_2}\right], \quad D_2 = k - f_{jr} + k' f_{j\theta}/k^2; \tag{1.4}$$

$$\frac{P - P_j}{1 + \gamma P_j} = \frac{2}{\gamma + 1}\left[\frac{D_2^2}{a_{j*}^2 (1 + k'^2/k^2)} - 1\right]; \quad f_r - f_{jr} = k'(f_{j\theta} - f_\theta); \tag{1.5}$$

$$r_* = f_{jr} - r_*' f_{j\theta}/r_*^2 + a_{j*}\sqrt{1 + r_*'^2/r_*^2}; \tag{1.6}$$

$$f_r = f_{jr};\ f_\theta = f_{j\theta};\ P = P_j \quad \text{for} \quad r = r_*. \tag{1.7}$$

Here $a_* = a/a_0$, and the velocity potential and the pressure of the j-th flow can be written in the form

$$f_j = \varepsilon\,[br\cos(\theta-\varphi)+e] = \varepsilon f_j^{(1)} + \varepsilon^2 f_j^{(2)} + \ldots; \tag{1.8}$$
$$P_j = (p_j - p_0)/\gamma p_0 = \varepsilon d_j + \varepsilon^2 l_j + \ldots,$$

where d_j, l_j, b, and e are known constants; φ is the angle between the axis $\theta=0$ and the normal to the plane shock wave initiating the j-th flow.

We obtain the equation of the weak discontinuity line by integrating (1.6) and using (1.8) ($b=b_1+\varepsilon b_2+\ldots$; $e=e_1+\varepsilon e_2+\ldots$; $\varphi=\varphi_1+\varepsilon\varphi_2+\ldots$):

$$r_* = \varepsilon b\cos(\theta-\varphi) + \sqrt{1-(\gamma-1)\varepsilon e - \varepsilon^2 b^2[\sin^2(\theta-\varphi)+(\gamma-1)/2]} = 1 + \varepsilon r_*^{(1)} + \varepsilon^2 r_*^{(2)} + \ldots, \tag{1.9}$$

where

$$r_*^{(1)} = b_1\cos(\theta-\varphi_1) - \frac{\gamma-1}{2}e_1;\; r_*^{(2)} = b_2\cos(\theta-\varphi_1) + b_1\varphi_2 \times$$
$$\times \sin(\theta-\varphi_1) - \frac{\gamma-1}{2}e_2 - \frac{b_1^2}{2}\sin^2(\theta-\varphi_1) - \frac{\gamma-1}{4}b_1^2 - \frac{(\gamma-1)^2}{8}e_1^2.$$

Equations (1.4)-(1.9) are greatly simplified for a front propagating in a stationary gas ($f_j = P_j = 0$).

The problem is to integrate the system of nonlinear equations (1.2) and (1.3) with the boundary conditions (1.4), (1.5), (1.7), and (1.9) and the conditions on the edges of the wedge.

We seek the solution in the form of asymptotic series in a small parameter:

$$P = \varepsilon P^{(1)} + \varepsilon^2 P^{(2)} + \ldots;\; f = \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \ldots; \tag{1.10}$$
$$k = 1 + \varepsilon k^{(1)}(\theta) + \varepsilon^2 k^{(2)}(\theta) + \ldots;\; r_* = 1 + \varepsilon r_*^{(1)} + \varepsilon^2 r_*^{(2)} + \ldots. \tag{1.11}$$

Substituting (1.10) into (1.2) and (1.3), we obtain the systems of equations for the first and second approximations. The appropriate boundary conditions are obtained by substituting (1.10) and (1.11) into (1.4), (1.5), and (1.7) and into the condition at the wedge boundaries. The perturbed regions in the linear approximation are shown in Fig. 1.

If we eliminate $f^{(1)}$ from the system of equations for the first approximation and apply the Busemann–Chaplygin transformation $\sigma=(1-\sqrt{1-r^2})/r$, $\theta=\theta$, the problem is reduced to finding the function $p^{(1)}(\sigma,\theta)$, which is harmonic in the perturbed region and has piecewise-constant values on the boundary $r=\sigma=1$ [the values of P(1) in Fig. 1 are shown alongside the fronts: 0, $\sin\alpha_1$, $\sin\alpha_2$, M_0, $m=M_0^2/\sqrt{M_0^2-1}$] and satisfies the condition $P_\theta^{(1)}=0$ on the boundaries of the wedge. In addition, in case (b) at point E with coordinates $\sigma=-\xi_0=-(1-\sqrt{1-M_0^2})/M_0$, $\theta=\pi$, the function $P^{(1)}$ must have a given singularity [1].

The solutions of the problems indicated have the form

$$P_1^{(1)} = \sin\alpha_1\cdot I\{\sigma^\lambda;\,\lambda(\theta-\alpha_1),\,-(\lambda/2)\pi\} + \sin\alpha_2\cdot I\{\sigma^\lambda,\,\lambda(\theta-\alpha_1),\,\lambda(3\pi/2-\alpha_1-\alpha_2)\}; \tag{1.12}$$
$$P_2^{(1)} = \frac{m}{\pi}\ln\sqrt{\frac{1-2\xi_0\sigma\cos\theta+\xi_0^2\sigma^2}{\sigma^2+2\xi_0\sigma\cos\theta+\xi_0^2}} + M_0 I\{\sigma,\,\theta,\,-\pi/2\};$$
$$P_3^{(1)} = mI\{\sigma,\,\theta,\,\pi/2+\arcsin(1/M_0)\} + M_0 I\{\sigma,\,\theta,\,-\pi/2\},\;\lambda=\pi/(2\pi-\alpha_1-\alpha_2),$$

where the function I is given by the relation

$$I\{s,\;w,\;w_1\} = (1/\pi)\mathrm{arctg}\{[(1-s^2)\sin w_1]/[2s\cos w-(1+s^2)\cos w_1]\}. \tag{1.13}$$

The values of arctan in (1.13) are taken in the interval $(0,\pi)$. The lines of equal pressure (isobars) calculated from (1.12) for certain values of the initial parameters (a) $\alpha_1=60°$, $\alpha_2=15°$; (b) $M_0=0.5$; $M_0=\sqrt{2}$ are shown in Fig. 1a-c, respectively.

In case (c) upward and downward flows from the wedge are independent of one another. For an asymmetric thin wedge $\alpha_2=A\alpha_1$, $0<A\le 1$ the pressure in the perturbed region below the wedge is $P_4^{(1)}=AP_3^{(1)}$ and the lift is directed downward.

Analysis of the results of the linear theory show that they must be refined near the boundaries of the perturbed region. From (1.12) as $r\to 1$

$$P_i^{(1)} = d_j + \sqrt{1-r}\,Q_i/(\pi\sqrt{2}) + 0\,[(1-r^{3/2})],\; i=1,2,3; \tag{1.14}$$
$$Q_1(\theta,\alpha_1,\alpha_2) = \lambda\left\{\sin\alpha_1\left[\mathrm{ctg}\frac{\lambda}{2}\left(\theta-\frac{\pi}{2}-\alpha_1\right)-\mathrm{ctg}\frac{\lambda}{2}\left(\theta+\frac{\pi}{2}-\alpha_1\right)\right]+\right.$$
$$\left.+\sin\alpha_2\left[\mathrm{ctg}\frac{\lambda}{2}\left(\theta+\frac{3\pi}{2}-2\alpha_1-\alpha_2\right)-\mathrm{ctg}\frac{\lambda}{2}\left(\theta-\frac{3\pi}{2}+\alpha_2\right)\right]\right\};$$

$$Q_2(\theta, M_0) = M_0\left[\operatorname{ctg}\left(\frac{\theta}{2} - \frac{\pi}{4}\right) - \operatorname{ctg}\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \frac{2\xi_0}{1 - 2\xi_0\cos\theta - \xi_0^2}\right];$$

$$Q_3(\theta, M_0) = m\left[\operatorname{ctg}\left(\frac{\theta}{2} + \frac{\pi}{4} + \frac{1}{2}\arcsin\frac{1}{M_0}\right) - \operatorname{ctg}\left(\frac{\theta}{2} - \frac{\pi}{4} - \frac{1}{2}\right.\right.$$
$$\left.\left.\times \arcsin\frac{1}{M_0}\right)\right] + M_0\left[\operatorname{ctg}\left(\frac{\theta}{2} - \frac{\pi}{4}\right) - \operatorname{ctg}\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right].$$

The pressure gradient in the neighborhood of $r=1$ clearly contradicts the assumptions on which the linear (acoustic) theory is based.

From an analysis of the exact equations and boundary conditions (1.2)-(1.9) similar to that performed in the study of nonlinear conical gas flows [9] it can be established that in the first approximation

$$P_r(k, \theta) = -2/(\gamma + 1) + \ldots;\; P_r(r_*, \theta) = 1/(\gamma - 1) + \ldots. \tag{1.15}$$

By analyzing the behavior of the functions Q_i and using (1.14) we can establish that the lines of the fronts AB and CD are weak discontinuity lines (along them $Q_i < 0$) and lines of the front BC are weak shock waves (along them $Q_i > 0$).

Let us determine the limits of applicability of the linear theory.

A. In the neighborhood of a weak shock front BC propagating in a stationary gas ($d_j = 0$) the asymptotic behavior of $f^{(1)}$ according to the linearized Lagrange–Cauchy integral (1.3) is written in the form

$$f^{(1)} = -(2/3)cQ(1 - r)^{3/2} + 0[(1 - r)^{5/2}],\; c = 1/(\pi\sqrt{2}), \tag{1.16}$$

where from now on we omit the subscript i.

Using (1.14), (1.16), (1.10), and (1.11) in (1.2) and (1.3) we obtain

$$P = \varepsilon cQ\sqrt{1 - r} + \varepsilon^2[(\gamma + 1)/2]c^2Q^2 + \ldots;$$
$$f = -\varepsilon cQ(1 - r)^{3/2} - \varepsilon^2[(\gamma + 1)/2]c^2Q^2(1 - r) + \ldots. \tag{1.17}$$

For $r \sim 1 - \varepsilon^2Q^2$ expansions (1.17), which are outer according to the terminology of [10], are irregular; i.e., the lower terms of the expansions become of the same order as the higher terms.

According to (1.17) the inner variables are determined by the expressions

$$r = 1 - \varepsilon^2Q^2\delta;\; P = [2\varepsilon^2/(\gamma + 1)]Q^2\Pi(\delta) + \ldots;$$
$$f = -[2\varepsilon^4/(\gamma + 1)]Q^4F(\delta) + \ldots. \tag{1.18}$$

The corresponding system of nonlinear equations, analogous to the equations of one-dimensional short waves [11], and their general solution have the form

$$2(F_\delta + \delta)F_{\delta\delta} - F_\delta = 0;\; \Pi = F_\delta;$$
$$\Pi = (1 \pm \sqrt{1 + \delta c_1})/c_1;\; F = \delta/c_1 \pm 2(1 + \delta c_1)^{3/2}/(3c_1^2) + c_2,$$

where c_1 and c_2 are arbitrary functions of θ.

The equation of the shock front is $\delta = \delta_*$, and the conditions on it are obtained from (1.18), (1.4), and (1.5):

$$k = 1 - \varepsilon^2Q^2\delta_*;\; F_\delta(\delta_*) = -2\delta_*;\; \Pi(\delta_*) = F_\delta(\delta_*);\; F(\delta_*) = 0. \tag{1.19}$$

Joining the outer expansion of order ε^2 to the inner expansion of order ε^2 and satisfying conditions (1.19), we obtain

$$P = f_r = \varepsilon^2[(\gamma + 1)/2]c^2Q^2(1 + D^{1/2}) + \ldots;$$
$$P_r = -1/(\gamma + 1)D^{1/2} + \ldots,\; D = 1 + 4(1 - r)/\varepsilon^2(\gamma + 1)^2c^2Q^2,$$
$$f = -[(\gamma + 1)/2]\varepsilon^2c^2Q^2(1 - r) - [(\gamma + 1)^3/12]\varepsilon^4c^4Q^4(1 + D^{3/2}) + \ldots;$$
$$k - 1 = [(\gamma + 1)/4]P(k, \theta) = (3/16)\varepsilon^2(\gamma + 1)^2c^2Q^2 + \ldots. \tag{1.20}$$

The expressions for the position of the shock front and the strength agree with those obtained by the method of strained coordinates [5].

B. In the neighborhood of a front of a weak discontinuity line propagating along the j-th flow, like case A, using the equations for the second approximation as $r \to 1$, we find

$$P = \varepsilon\left\{d_j + cQ\sqrt{1 - r} + 0\left[(1 - r)^{3/2}\right]\right\} + \varepsilon^2\left\{l_j + r_*^{(1)}cQ/2\sqrt{1 - r} + 0\left(\sqrt{1 - r}\right)\right\} + 0(\varepsilon^3),$$
$$f = \varepsilon\left\{f_j^{(1)} - (2/3)cQ(1 - r)^{3/2} + 0\left[(1 - r)^{5/2}\right]\right\} + \varepsilon^2\left\{f_j^{(2)} - r_*^{(1)}cQ\sqrt{1 - r} + 0\left[(1 - r)^{3/2}\right]\right\} + 0(\varepsilon^3). \tag{1.21}$$

The expansions (1.21) are irregular for $1-r\sim\varepsilon$, and therefore it is necessary to introduce the inner variables

$$r = 1 - \varepsilon z;\; f = f_j - [2/(\gamma+1)][\varepsilon^{5/2}G_1(z,\ \theta) + \varepsilon^3 G_2(z,\ \theta) + \ldots];$$
$$P = P_j + [2/(\gamma+1)][\varepsilon^{3/2}\Pi_1 + \varepsilon^2\Pi_2 + \ldots].$$

The equations for the first two approximations

$$\Pi_1 = G_{1z},\; 2\left(z + r_*^{(1)}\right)G_{1zz} - G_{1z} = 0;$$
$$\Pi_2 = G_{2z};\; 2\left(z + r_*^{(1)}\right)G_{2zz} - G_{2z} + 2G_{1z}G_{1zz} = 0$$

are easy to solve (the c_i are arbitrary functions of θ):

$$P = P_j + \frac{2}{\gamma+1}\left\{\varepsilon^{3/2}c_3\sqrt{z + r_*^{(1)}} + \varepsilon^2\left(c_5\sqrt{z + r_*^{(1)}} + c_3^2\right) + \ldots\right\}; \tag{1.22}$$
$$f = f_j - \frac{2}{\gamma+1}\left\{\left[\frac{2c_3}{3}\left(z + r_*^{(1)}\right)^{3/2} + c_4\right]\varepsilon^{5/2} + \varepsilon^3\left[\frac{2c_5}{3}\left(z + r_*^{(1)}\right)^{3/2} + c_3^2 z + c_6\right] + \ldots\right\}.$$

Joining (1.21) and (1.22), we find $c_3 = [(\gamma+1)/2]cQ$, $c_4 = c_5 = 0$. Expansions (1.22) must also satisfy the inner boundary conditions following from (1.9) and (1.7). It may be noted, however, that the expansions are not uniformly applicable in the neighborhood of a weak discontinuity line and satisfy the boundary conditions only up to quantities of order ε.

The middle expansions (1.22) become irregular for $1-r+\varepsilon r_*^{(1)}\sim\varepsilon^2 c_3^2$. Using these expansions we transfer the singularity from the line $r=1$ to the line $r=1+\varepsilon r_*^{(1)}$.

In accordance with (1.22) we write the inner expansion in the form

$$r = 1 + \varepsilon r_*^{(1)} - \varepsilon^2 c_3^2\omega;\; P = P_j + \frac{2}{\gamma+1}\varepsilon^2 c_3^2\Pi_3(\omega) + \ldots; \tag{1.23}$$
$$f = f_j - \frac{2}{\gamma+1}\varepsilon^4 c_3^4 G_3(\omega) + \ldots .$$

The nonlinear equations for G_3 and Π_3, which describe the flow close to a front propagating by a uniform stream, and their general solution have the form

$$2G_{3\omega\omega}\left(\omega + G_{3\omega} + r_*^{(2)}/c_3^2\right) - G_{3\omega} = 0;\; \Pi_3 = G_{3\omega}; \tag{1.24}$$
$$\Pi_3 = \left(1 \pm \sqrt{1 + c_7\left(\omega + r_3^{(2)}/c_3^2\right)}\right)/c_7;$$
$$G_3 = \left(\omega + r_*^{(2)}/c_3^2\right)/c_7 \pm \frac{2}{3c_7^2}\left[1 + c_7\left(\omega + r_*^{(2)}/c_3^2\right)\right]^{3/2} + c_8,$$

where $c_3 = [(\gamma+1)/2]cQ$, c_7, c_8 are arbitrary functions of θ. Matching (1.23) and (1.24) and the middle expansions (1.22) we determine $c_7=1$. The solution (1.23) and (1.24) with the lower signs and $c_8=2/3$ describes the flow behind a weak discontinuity line and satisfies the conditions (1.7) to terms of order ε^2 inclusively.

The expressions for the pressure gradients calculated from the inner solutions (1.20), (1.23), and (1.24) agree with (1.15).

We note an important detail inherent in the problem and in many respects determining the effectiveness of the plots drawn: the inner variables δ and ω depend on r and θ.

The equation of the lines of equal pressure $P^{(1)}=p_*$ in the neighborhood of a weak discontinuity front can be obtained from (1.24) and (1.23):

$$r = 1 + \varepsilon r_*^{(1)} + \varepsilon^2 r_*^{(2)} + (\gamma+1)\varepsilon(p_* - d_j) - (p_* - d_j)^2/(c^2Q^2). \tag{1.25}$$

The equation of the lines $P^{(1)}=p_*$ in the neighborhood of a shock front is obtained from (1.20) and has the form of (1.25) with $r_*^{(1)} = r_*^{(1)} = d_j = 0$. Calculations showed that close to the fronts there is an essential difference between the actual behavior of the parameters and the linear theory predictions.

C. We refine the position of the plane shock fronts. The equation of the front BB_1 in cases a–c (Fig. 1a–c) is

$$r\cos(\theta - \alpha_1) = [(\gamma+1)/4]M_0\sin\alpha_1 + \sqrt{1 + \{[(\gamma+1)/4]M_0\sin\alpha_1\}^2}. \tag{1.26}$$

The equation of the front ED in case (c) is

$$M_0\sin\beta = r\sin(\theta - \beta), \tag{1.27}$$

where β is the angle between the front ED and the axis $\theta = 0$ for which the relation [4]

$$\beta = \arcsin(1/M_0) + [(\gamma+1)/4]\left[M_0^2/(M_0^2-1)\right]\alpha_1 + 0(\alpha_1^2)$$

holds.

§ 2. The limits of applicability of the theory of Sec. 1, henceforth referred to as the one-dimensional inner theory, are determined by the possibility of representing linear solutions (1.12) near $r=1$ in the form (1.14). It can be shown that the conditions are violated for $|\theta - \theta_*| \sim \sqrt{1-r}$, where θ is the angle of any point of intersection of three fronts (points B, C of Fig. 1). Taking account of the fact that $Q_i \sim 1/(\theta - \theta_*)$ as $\theta \to \theta_*$, it can be concluded from (1.18) and (1.23) that the one-dimensional inner theory is inapplicable when

$$r \sim 1 - 0(\varepsilon),\ |\theta - \theta_*| \sim 0(\varepsilon^{1/2}). \tag{2.1}$$

To investigate the flow in the interaction zones whose dimensions are determined by (2.1) it is necessary to introduce inner variables [according to (2.1), (1.18), and (1.23) in the interaction zones $P \sim \varepsilon$, $f \sim \varepsilon^2$]

$$r = 1 - [(\gamma+1)/2]E\Delta;\ \theta_* - \theta = \sqrt{[(\gamma+1)/2E}Y; \tag{2.2}$$
$$P = E\mu(\Delta, Y) + \ldots;\ f = -[(\gamma+1)/2]E^2G(\Delta, Y) + \ldots.$$

The positive constants in (2.2) were chosen to simplify the form of the inner equations and the boundary conditions for the interactions considered. For the interaction regions occurring in cases (a) $E = M_0 \sin\alpha_j$; (b) $E = \alpha M_0$; (c) $E = \alpha M_0$, $\alpha M_0^2/\sqrt{M_0^2-1}$.

An important consequence of (2.2) is the similarity rule: the flows in the interaction zones in problems with various initial values of M_0 and α_j are similar if the similarity parameter E is the same for these values. Substituting (2.2) into (1.2) and (1.3) and introducing the notation $\nu = G_Y$, we obtain a system of nonlinear equations describing the flow in the interaction zone in the first approximation:

$$2(\Delta + \mu)\mu_\Delta - \mu + \nu_Y = 0;\ \mu = G_\Delta;\ \nu_\Delta = \mu_Y. \tag{2.3}$$

We note that similar systems of equations are obtained in investigating the interaction of weak shock waves [6, 7, 8, 12, 13] and conical gas flows [9, 14, 15].

In the cases considered the interaction zones are of the same type and the construction of solutions in them can be reduced to the solution of the following boundary-value problem: find the solution of the system of equations (2.3) in the region $BB_2B_3B_4B_5B$ (Fig. 2) which pass over continuously along BB_5 into the known solution in the region B_1BB_5 satisfying the inner conditions on BB_2 and the matching conditions with the known solutions on $B_2B_3B_4B_5$.

The position of plane fronts BB_1 (1.26), (1.27) in the variables (2.2) is written in the form

$$2\Delta + Y^2 = -1,\ \Delta \leqslant -1,\ |Y| \geqslant 1, \tag{2.4}$$

and the position of the weak discontinuity line BB_5 in the form

$$\Delta = -1;\ |Y| \geqslant 1. \tag{2.5}$$

Solving (2.4) and (2.5) simultaneously, we determine the coordinates of point B:

$$\Delta_B = -1;\ |Y_B| = 1. \tag{2.6}$$

Henceforth, we present the formulation of the boundary-value problem only for the case $\theta_B < \theta_*$ (Fig. 2a); in Eqs. (2.4)-(2.6) $Y > 0$.

For the damped shock front BB_2 (Fig. 2b) we obtain from the exact relations (1.4) and (1.5)

$$d\Delta/dY = -\sqrt{-2\Delta - \mu};\ d\Delta/dY = -\nu/\mu; \tag{2.7}$$
$$\mu = G_\Delta\ (-1 \leqslant \Delta < 0,\ -\infty < Y \leqslant 1,\ \mu \leqslant -2\Delta).$$

The last of Eqs. (2.7) is satisfied as a result of (2.3).

The matching conditions are the inner expansions of order ε^2 and the corresponding outer (with respect to the interaction zone) expansions of order ε^2. Taking account of higher order terms in the outer expansions does not change the matching conditions.

By requiring the position of the front BB_2 to agree with the position given by the one-dimensional inner theory [the last of Eqs. (1.20)] we find the equation of the front BB_2 in the neighborhood of point B_2 and the values of the parameters on it as $Y \to -\infty$:

$$\Delta = -3/2\pi^2Y^2 + \ldots;\ \mu = 3/\pi^2Y^2 + \ldots; \tag{2.8}$$
$$\nu = -9/\pi^4Y^5 + \ldots .$$

On the boundary B_2B_3 the conditions for joining with the solution (1.20) as $V \to -\infty$, $\Delta \geqslant -3/2\pi^2Y^2$:

$$\mu = (2/\pi^2Y^2)(1 + D_*^{1/2}),\ D_* = 1 + (\pi^2Y^2/2)\Delta;$$
$$\nu = -(4\Delta/\pi^2Y^3)(1 - D_*^{1/2}) - (32/3\pi^4Y^5)(1 + D_*^{3/2}). \tag{2.9}$$

Using (1.10) and (1.12) the conditions on the boundary B_3B_4 as $\Delta \to \infty$, $-\infty < Y < \infty$ are

$$\mu = (1/\pi)\ \mathrm{arctg}(\sqrt{2\Delta} - Y); \tag{2.10}$$
$$\nu = (1/\pi)(Y\ \mathrm{arctg}(\sqrt{2\Delta} - Y) + \sqrt{2\Delta}).$$

The values of arctan in (2.10) are taken in the interval $(0, \pi)$.

On the boundary B_4B_5 the conditions for joining with the solutions (1.23) and (1.24) as $Y \to \infty$, $\Delta \geq -1$ are

$$\mu = 1 + (2/\pi^2Y^2)(1 - D_{1*}^{1/2}),\ D_{1*} = 1 + (\pi^2Y^2/2)(\Delta + 1); \tag{2.11}$$
$$\nu = Y - (4(\Delta + 1)/\pi^2Y^3)(1 + D_{1*}^{1/2}) - (32/3\pi^4Y^5)(1 - D_{1*}^{3/2}).$$

According to (1.7) and (1.8) on the weak discontinuity line BB_5 for $\Delta = -1$, $Y \geq 1$

$$\mu = 1;\ \nu = Y. \tag{2.12}$$

We note that in (2.9)-(2.12) only one of the conditions for μ and ν (either) need be retained, since the conditions satisfy the second equation of the system (2.3).

Thus, the problem (2.3), (2.7)-(2.12) is a boundary-value problem for a system of nonlinear elliptic equations with an unknown element of the boundary [the position of the front BB_2 is determined in terms of the solution $\mu = \mu(\Delta, Y)$] and its solution describes in the first approximation the flow in the interaction zones arising in the problems considered.

§ 3. Let us construct an approximate solution of problem (2.3), (2.7)-(2.12). The exact particular solution of the system of equations (2.3) which is analogous to that found in [11]

$$\Delta = (Y^2/2)\,\mathrm{tg}^2(\mu k_1 + k_2) - \mu - (1/2k_1)\sin 2(\mu k_1 + k_2) + B\sin^2(\mu k_1 + k_2); \tag{3.1}$$
$$\nu = Y[\mu - (1/k_1)\,\mathrm{tg}(\mu k_1 + k_2)],\ k_1, k_2, B - \mathrm{const},$$

satisfies the conditions (2.9)-(2.12) and conditions (2.7) at point B for the following choice of constants:

$$k_1 = -k_2 = \pi \text{ for } Y > 0;\ k_1 = \pi,\ k_2 = 0 \text{ for } Y < 0. \tag{3.2}$$

We note that a different choice of constants for $Y < 0$ and $Y > 0$ does not imply that solution (3.1) is discontinuous at $Y = 0$ ($-\tan\pi(1 - \mu) = \tan \pi\mu$ etc.).

When conditions (2.7) and (2.8) are satisfied there is only one arbitrary constant B. If we transform to the new independent variables $\mu\nu$, Eq. (2.7) takes the form

$$d\nu/d\mu = -(\Delta_\mu + Y_\mu\sqrt{-2\Delta - \mu})/(\Delta_\nu + Y_\nu\sqrt{-2\Delta - \mu}); \tag{3.3}$$
$$d\nu/d\mu = -(\mu\Delta_\mu + \nu Y_\mu)/(\mu\Delta_\nu + \nu Y_\nu).$$

Direct substitution shows that the family of curves

$$\nu^2 = [\pi^2\mu^2 + \pi\sin 2\pi\mu(\mu + B/2) - \sin^2\pi\mu + B\pi^2(1 - \mu\cos 2\pi\mu)]/k_3, \tag{3.4}$$
$$k_3 = \pi^2(\mu - (1/2\pi)\sin 2\pi\mu)/[\cos\pi\mu(1 - (1/\pi)\mathrm{tg}\pi\mu)]^2$$

satisfies the second of conditions (3.3). The choice from (3.4) of the curve having the correct asymptotic behavior as it emerges from the interaction zones, as determined by Eqs. (2.8), leads to the requirement $B = 0$.

The remaining relation of (3.3) is the differential equation of the front BB_2, and if solution (3.1), taking account of (3.2) and the fact that $B = 0$, is an exact solution of problem (2.3), (2.7)-(2.12), the curve determined by it [we have the initial conditions for the integration from (2.6) and (2.12)] should agree with curve (3.4) for $B = 0$. Unfortunately, this is not the case, but calculations showed that the difference in position of these curves is very small.

From the solution (2.2), (3.2), and (3.4) we can construct the distribution of isobars in the interaction zone in the Δ, Y plane and in the physical plane for various values of the similarity parameter E. These dis-

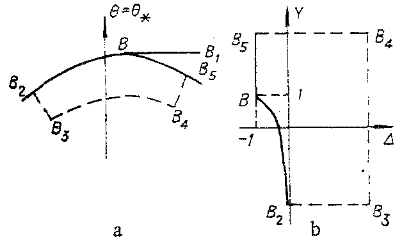

Fig. 2

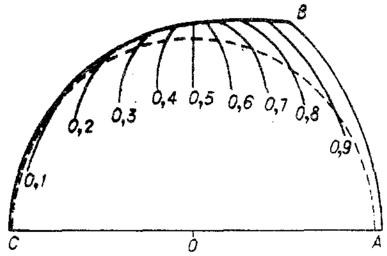

Fig. 3

tributions practically coincide with those constructed from (2.2), (3.1), (3.2), and the first of Eqs. (3.3), and, consequently, will be a good approximation to the exact solution of the boundary-value problem (2.3), (2.7)-(2.12).

Figure 3 shows the distribution of isobars in the physical plane for $E = 0.175$. The open curve indicates the boundary of the perturbed region in the linear approximation. Beyond the limits of the interaction zone the level lines are calculated from (1.25). The only calculations available for comparison with our results are those performed numerically in [3] for case (b). Comparison shows satisfactory agreement between the results of the nonlinear theory and the numerical calculations ($E = 0.175$, Fig. 3 of [3]). Outside the neighborhood of the wave boundary of the perturbed region the nonlinear solutions agree with those of the linear theory (1.12), and for $E < 0.04$ they go over practically continuously into (1.12).

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